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Novel integrable cases in the dynamics of a body interacting with a medium taking into account dependence of the moment of the resistance force on the angular velocity☆

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ABSTRACT

Planar and spatial non-linear models of the action of a medium on a rigid body which take into account the dependence of the arm of the force on the reduced angular velocity of the body are constructed, when the moment of the force is also a function of the angle of attack. New cases of complete integrability in terms of elementary functions are found, enabling of qualitative similarities to be discovered between the motions of free bodies in a resistive medium and the oscillations of bodies that are partially fixed and are immersed in a flow of the medium. It is shown that if the additional damping action of the medium on the body that appears in the system is significant, the attainment of stability of the rectilinear translational deceleration of the body is possible when it moves with finite angles of attack. The question of the roughness of the description of this phenomenon is of current interest: a finer property of relative roughness is discovered when reduced dynamical systems are investigated. ©2008.

In the problem considered here, a body interacts with a medium only through the flat front portion (plate) of its surface. The force field is constructed using information on the jet flow properties under quasistationary conditions. The motion of the medium is not studied, and the problem of the dynamics of a rigid body under which the characteristic time of the motion of the body relative to its centre of mass is commensurate with the characteristic time of the motion of the centre of mass is considered.

From a practical point of view, it is important to investigate the stability of rectilinear translational (unperturbed) motion, under which the velocities of points in a body are perpendicular to a plate. At the initial stage of such an investigation, the dependence of the moment of the force exerted by the medium on the angular velocity of the body was neglected, and only the dependence on the angle of attack was taken into account. The results obtained indicate the impossibility of finding conditions under which the corresponding non-linear systems have solutions that describe the angular oscillations of the body of constrained amplitude, which are of primary interest.

The experiment on the motion of uniform circular cylinders in water in Ref. 1[1] confirmed that when the interaction of a body with a medium is modelled, the dependence of the moment of the force exerted by the medium on the angular velocity of the body must, in fact, be taken into account. When this is done, additional terms that introduce dissipation into the system appear in the equations of motion.

1. Statement of the problem of the plane-parallel motion of a symmetrical rigid body in a resistive medium

Suppose a uniform rigid body of mass m undergoes rectilinear and translational motion in a medium and the front portion of the body surface is a flat plate (segment AB in Fig. 1) that is under the conditions of jet flow. This means that when there are no sheat forces, the

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[1] See also: Samsonov VA, Shamolin, MV, Yeroshin VA, Makarshin VB. Mathematical modelling in the problem of the deceleration of a body in a resistive medium in jet flow. *Nauchn Otchet Inst Mekh Mosk Gos Univ* 1995;(4396).

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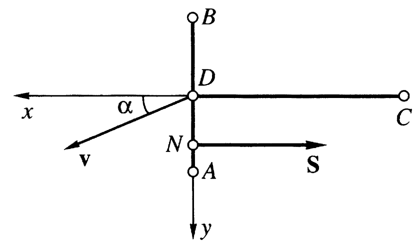

Fig. 1.

action of the medium on the body (plate) reduces to the force **S** (applied at point N), which is orthogonal to the plate (Fig. 1). The remainder of the body surface can be positioned within the volume bounded by the jet surface extending from the edge of the plate, and does not experience the action of the medium. Similar conditions can appear, for example, after entry of a body into water.[2] It is also assumed that the gravitational force is negligibly small compared with the resistance force of the medium.

We associate the right-hand system of coordinates $Dxyz$ with the plate (the z axis is perpendicular to the plane of the figure) and, for of simplicity, we will assume that Dzx is a plane of symmetry of the body. Then a regime of rectilinear translational (unperturbed) motion perpendicular to plate AB exists. The middle perpendicular Dx dropped from the centre of gravity C of the body onto the plane of the plate belongs to the line of action of the force **S**, and when this regime is disturbed, the velocity vector υ of point D deviates from the axis of symmetry by a certain angle (the angle of attack).

To construct a dynamical model, we will introduce the first three phase coordinates: υ, which is the magnitude of the velocity of point D relative to the flow (Fig. 1), α, which is the angle of attack, and Ω, which is the algebraic value of the projection of the absolute angular velocity of the body onto the z axis. We also have $AB = \Delta$.

We will assume that the magnitude of the force **S** depends on the square of υ with a certain coefficient s_1: $S = s_1\upsilon^2$ (the Newtonian resistance). It is usually assumed that $s_1 = \rho P c_x/2$, where ρ is the density of the medium and P is the area of the plate. The dimensionless drag coefficient c_x depends on the angle of attack, the Strouhal number, and other quantities, which are usually regarded as parameters in static models. In what follows, we will introduce the dimensionless "Strouhal-like" phase variable $\omega = \Omega\Delta/\upsilon$, as well as the auxiliary function $s = s_1 \operatorname{sign} \cos\alpha$. The action of the medium on the body will be specified by two functions: y_N and s.

We will confine ourselves to the case when of c_x depends only on the angle of attack, i.e., we will assume that s is a function of α and that $y_N = DN$ is a function of the two dimensionless variables α and ω.

Unlike the preceding studies,[3] henceforth the problem of the motion of a body will be studied in a non-linear formulation in the case when s depends on the angle of attack and also depends y_N on the reduced angular velocity ω.

The conditions $\alpha(t) \equiv 0$ and $\omega(t) \equiv 0$ hold for unperturbed plane-parallel motion; therefore, for small α and ω we take the function $y_N(\alpha, \omega)$ in the form

$$y_N = \Delta(k\alpha - h\omega) \tag{1.1}$$

where k and h are certain constants. We will neglect the dependence of s on α by virtue of the geometric symmetry of the body, which ensures that s is an even function. Therefore, the linearized model of the force exerted by the medium contains three parameters ($s = s_1, k, h$), which are determined by the shape of the plate in the scheme. The first of these parameters is dimensional, while k and h are dimensionless by virtue of the way in which they are introduced.

The values of s and k can be determined experimentally by means of weight measurements in systems of the hydro- or wind-tunnel tube type. In the literature there is also information[3,4] on the theoretical determination of these parameters for individual plate shapes, which allows us to assume that $k > 0$. The parameter h introduces a dependence of the moment of force on the angular velocity into the system, but the need to introduce it into the model is not obvious.

The study of the properties of the motion of the classes of bodies under consideration at the Institute of Mechanics of the M. V. Lomonosov Moscow State University was begun with experiments that were concerned with recording the motion of uniform circular cylinders in water. The experiments (whose results we processed) provide evidence that the rectilinear translational deceleration of a body (in water) is unstable, at least with respect to the angle of orientation of the body. In these experiments, it was possible to determine the parameters k and h of the action of the medium on a body.[1] It was also found that an additional parameter, which is equivalent to the so-called rotational derivative of the moment of hydro- or aerodynamic forces with respect to the angular velocity of the body, must be taken into account when modelling the action of a medium on a body. This parameter introduces additional dissipation into the system. On a purely formal level, stability of the unperturbed motion can be achieved by increasing its value. In certain media (for example, in clay) the motion is, in fact, stable in the sense described above.[5] Such stability is possibly achieved owing to the occurrence of considerable damping exerted by the medium in the system or the presence of forces that are tangential to the plate.

We henceforth single out a more general class of problems, under which a follower force (tractive force) **T** is applied along the straight line CD (Fig. 1) along with the force exerted by the medium, under the same assumptions regarding the nature of the interaction of the body with the medium. One such problem has already been solved[3] under the condition of constant traction, and instability of the unperturbed motion was demonstrated.

We note cases of motion which were subsequently subjected to thorough analysis:

1) the free deceleration of a body, i.e., the motion of a body under the action solely of the force exerted by the medium when there is no follower force;

2) the motion of a body under which the magnitude of the velocity of the centre of the plate is constant at all times during the motion (the presence of an unintegrable relation):

$$\upsilon \equiv \mathrm{const} \tag{1.2}$$

We will now obtain the equations of plane-parallel motion. We assign the position of the body in the plane by the coordinates (x_0, y_0) of point D and the angle of deviation φ from the axis of a certain inertial system of coordinates. The polar coordinates (υ, α) of the tip of the velocity vector of point D and the algebraic value of the projection of the angular velocity Ω are related to the variables $(\dot{x}_j, \dot{y}_0, \dot{\varphi}, \varphi)$ by the (unintegrable) kinematic relations

$$\dot{\varphi} = \Omega,\ \ \dot{x}_0 = \upsilon\cos(\alpha+\varphi), \quad \dot{y}_0 = \upsilon\sin(\alpha+\varphi) \tag{1.3}$$

We will define the phase state of the system in terms of the functions $(\upsilon, \alpha, \Omega, x_0, y_0, \varphi)$, and we will treat the first three of them as quasivelocities.

The kinetic energy of the body and the force exerted by the medium do not depend on the position of the body in the plane. This suggests that the coordinates (x_0, y_0, φ) are cyclic and leads to a reduction of the order of the overall system of equations of motion.

The equations of motion of the centre of mass in the Dxy axes and the variations of the angular momentum in König's axes form a closed system of differential equations, which can be treated in a three-dimensional phase space of quasivelocities ($\sigma = DC$, I is the central moment of inertia):

$$\dot{\upsilon}\cos\alpha - \dot{\alpha}\upsilon\sin\alpha - \Omega\upsilon\sin\alpha + \sigma\Omega^2 = -s(\alpha)\upsilon^2/m$$

$$\dot{\upsilon}\sin\alpha + \dot{\alpha}\upsilon\cos\alpha + \Omega\upsilon\cos\alpha - \sigma\dot{\Omega} = 0$$

$$I\dot{\Omega} = y_N(\alpha, \omega)s(\alpha)\upsilon^2; \quad \omega = \Omega\Delta/\upsilon \tag{1.4}$$

Systems (1.3) and (1.4) together form a complete sixth-order system for describing the plane-parallel motion of a rigid body in a medium under the action of the resistance force under quasistationary conditions. If the class of problems concerned with the motion of a body in the presence of a follower force introduced above is considered, the quantity $(T - s(\alpha)\upsilon^2)/m$ will appear on the right-hand side of the first equality in (1.4).

In particular, to ensure that condition (1.2) holds, it is sufficient to select the algebraic value T of the follower force in the following manner:

$$T = T(\upsilon, \alpha, \Omega) = m\sigma\Omega^2 + \upsilon^2[s(\alpha) - (m\sigma y_N(\alpha, \omega)s(\alpha)\sin\alpha)/(I\cos\alpha)]$$

The first equality in (1.4) is then satisfied identically. Note that case 2 has methodical value, since it enables us to subsequently reduce the order of the system of equations of motion (see also Ref. 6) and to obtain an important mechanical analogy.

2. Classes of functions that specify the action of the medium

The functions $y_N(\alpha, \omega)$ and $s(\alpha)$, which specify the action of the medium on the body, appear in the dynamical system (1.4). The function y_N (compare with expression (1.1)) depends not only on the angle of attack α, but also on the reduced angular velocity ω. If, in particular, the latter dependence is neglected (as it was in the so-called simplest assumption in the preceding studies), the quantity y_N is a function of only the angle of attack, i.e., $y_N = y(\alpha)$, and its dependence on the single argument is determined using experimental information on the jet flow properties.[4,7]

However, the main purpose of this paper is to take into account the influence of the rotational derivatives of the force exerted by the medium with respect to the angular velocity of the body, which requires the introduction of additional arguments, viz., the components of the angular velocity of the body, into the action functions of the medium. This is not a trivial modelling problem. As has already been noted, in this paper we will confine ourselves to introducing the angular velocity as an argument only into the function y_N, and we will neglect its analogous introduction into the reduced drag coefficient s.

By analogy with (1.1), we will consider y_N in the form

$$y_N(\alpha, \omega) \cong y_N(\alpha, \Omega/\upsilon) = y(\alpha) - H\Omega/\upsilon$$

Here $H > 0$ by virtue of the results of the experiment in Ref. 1, and the third equality in (1.4) takes the form

$$I\dot{\Omega} = F(\alpha)\upsilon^2 - Hs(\alpha)\Omega\upsilon; \quad F(\alpha) = y(\alpha)s(\alpha) \tag{2.1}$$

The system composed of the first two equalities in (1.4) and Eq. (2.1) contains the functions $F(\alpha)$ and $s(\alpha)$, whose explicit form is difficult to describe analytically even for plates of simple shape. Therefore, the technique of "embedding" this problem into a broader class of problems that takes into account only the qualitative properties of the functions $F(\alpha)$ and $s(\alpha)$ is used.

Construction of the functional classes $\{y\}$ and $\{s\}$ is facilitated by the result reported by Chaplygin, who obtained the functions $y(\alpha)$ and $s(\alpha)$ analytically for plane-parallel jet flow past a plate of infinite length:[8]

$$y(\alpha) = y_0(\alpha) = A\sin\alpha \in \{y\}, \quad A > 0; \quad s(\alpha) = s_0(\alpha) = B\cos\alpha \in \{s\}; \quad B > 0 \tag{2.2}$$

Combining relations (2.2) with experimental information on the properties of jet flow,[4,7] we formally describe these classes, which consist of fairly smooth, 2π-periodic functions ($y(\alpha)$ is an odd function, and $s(\alpha)$ is an even function) that satisfy the following conditions:

$y(\alpha) > 0$ when $\alpha \in (0, \pi)$, $y'(0) > 0$, and $y'(\pi) < 0$ (the class of functions $\{y\} = Y$); $s(\alpha) > 0$ when $\alpha \in (0, \pi/2)$, $s(\alpha) < 0$ when $\alpha \in (\pi/2, \pi)$, $s'(0) > 0$, and $s'(\pi/2) < 0$ (the class of functions $\{s\} = \Sigma$). Both y and s change sign when α is replaced by $\alpha + \pi$. Thus, $y \in Y$, and

$$s \in \Sigma \tag{2.3}$$

It follows from the above conditions that the function F introduced into (2.1) is a fairly smooth π-periodic odd function that satisfies the following conditions: $F(\alpha) > 0$ when $\alpha \in (0, \pi/2)$, $F'(0) > 0$, and $F(\pi/2) < 0$ (the class of functions $\{F\} = \Phi$). Thus, $F \in \Phi$.

In particular, the following analytic function will be a typical representative of the class of functions Φ (Ref. 8)

$$F = F_0(\alpha) = AB\sin\alpha\cos\alpha \in \Phi \tag{2.4}$$

The following question arises in connection with the previously noted instability of rectilinear translational deceleration.[1] Do angular oscillations of the axis of symmetry of the body of finite (constrained) amplitude exist?

We will formulate this question in a more general form. Does a pair of action functions y and s of the medium exist such that the constraint $0 < \alpha(t) < \alpha^* < \pi/2$ holds for a certain solution of the dynamical part of the equations of motion beginning at a certain time $t = t_1$?

It was previously shown,[9] under a very simple assumption regarding the properties of the functions y_N and s, that in a quasistationary description of this interaction (where y_N and s depend only on the angle of attack (i.e., $H = 0$)) there are no non-trivial constrained solutions in the system for any permissible pair of the functions $y(\alpha)$ and $s(\alpha)$ (or $F(\alpha)$) that specify the action of the medium over the entire range of finite angles of attack ($0 < \alpha < \pi/2$). Thus, for a possible positive answer to the question, we must enlist the dependence of the moment of the force exerted by the medium on the reduced angular velocity. As will be shown, a positive answer to this question can be expected under certain assumptions.

Of course, from a practical point of view, an analysis of the dynamical equations is important only in the vicinity of unperturbed motion, since the model of the action of a medium on a body under consideration ceases to be reliable at certain (critical) angles of attack.

Thus, plane-parallel jet flow around a plate is investigated using classes of dynamical systems defined using a pair of action functions of the medium, which considerably complicates the performance of the qualitative analysis.

3. The motion of a body in a resistive medium in the presence of a follower force

We consider the class of motions 2 described above, i.e., motions when there is a follower force acting the body that ensures that condition (1.2) is satisfied at all times. Then the positive parameter υ is added to the previously introduced parameters of the system.

Thus, the dynamical part of the equations of such motion of a rigid body in the presence of a follower force in case (1.2) reduces to the second-order system

$$\alpha^{\cdot}\upsilon\cos\alpha + \Omega\upsilon\cos\alpha - \sigma\Omega^{\cdot} = 0, \quad I\Omega^{\cdot} = F(\alpha)\upsilon^2 - Hs(\alpha)\Omega\upsilon, \quad H > 0$$

Outside and only outside the union of the straight lines

$$O = \{(\alpha, \Omega) \in R^2 : \alpha = \pi/2 + \pi k, \ k = 1, 2\} \tag{3.1}$$

which lie in the $R^2\{\alpha, \Omega\}$ plane, system (3.1) is equivalent to the system of normal form

$$\alpha^{\cdot} = -\Omega + \frac{\sigma\upsilon}{I}\frac{F(\alpha)}{\cos\alpha} - \frac{\sigma}{I}H\frac{s(\alpha)}{\cos\alpha}\Omega, \quad \Omega^{\cdot} = \frac{\upsilon^2}{I}F(\alpha) - H\frac{\upsilon}{I}\Omega s(\alpha) \tag{3.2}$$

We call system (3.2) a reference system when conditions (2.2) (or (2.4)) hold. After introducing the dimensionless parameters $\beta = \sigma^2 AB/I \geq 0$ and $H_1 = \sigma BH/I \geq 0$, as well as the dimensionless variable $\omega = \Omega\sigma/\upsilon$ and the new independent variable $\sigma_1 = \upsilon t/\sigma$, differentiation with respect to which we denote by a prime, the reference system takes the following form in the new notation:

$$\alpha' = -(1 + H_1)\omega + \beta\sin\alpha, \quad \omega' = [\beta\sin\alpha - H_1\omega]\cos\alpha \tag{3.3}$$

As was already indicated,[9] a non-linear autonomous second-order system that has asymptotic limit sets rarely has a first integral expressed in terms of a finite combination of elementary functions. In addition, such a first integral must be a transcendental function, i.e., it must have singular points (which correspond to the existing asymptotic attracting or repelling limit sets).

We next obtain the first integral for reference system (3.3) in explicit form. For this purpose, we compare this system with the equation

$$\frac{d\omega}{d\alpha} = \frac{\beta\sin\alpha\cos\alpha - H_1\omega\cos\alpha}{-\omega + \beta\sin\alpha - H_1\omega}$$

which, by making the substitutions $\tau = \sin\alpha$ and $\omega = \xi\tau$, can be brought to the form

$$[-\xi + \beta - H_1\xi][\xi d\tau + \tau d\xi] = [\beta - H_1\xi]d\tau; \ d\omega = \xi d\tau + \tau d\xi;$$

$$[-\omega + \beta\tau - H_1\omega]d\omega = [\beta\tau - H_1\omega]d\tau$$

The variables are easily separated, enabling us to obtain the quadrature formula

$$\int\frac{d\tau}{\tau} = \int\frac{[(H_1 + 1)\xi - \beta]d\xi}{-\xi^2(1 + H_1) + (\beta + H_1)\xi - \beta} \tag{3.4}$$

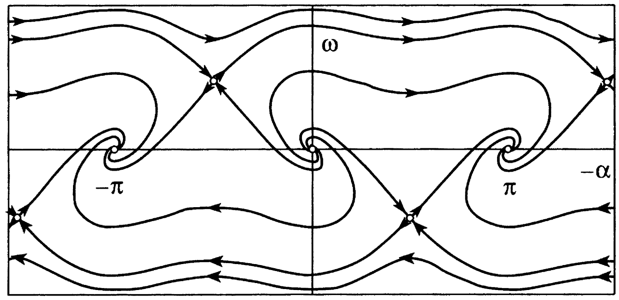

Fig. 2.

Thus, system (3.3) has a transcendental first integral, which is expressed in terms of elementary functions, since the first integral of system (3.3) is specified in explicit form using quadrature formula (3.4). Depending on the coefficients β and H_1, quadrature formula (3.4) can be represented in three different forms, which correspond to foci, nodes or degenerate (attracting or repelling) nodes that lie on the phase cylinder of system (3.3).

System (3.3) reduces to the equation of a reduced pendulum

$$\alpha'' + (H_1 - \beta)\alpha'\cos\alpha + \beta\sin\alpha\cos\alpha = 0 \tag{3.5}$$

When $H_1 > \beta$ ($H_1 < \beta$), the trivial solution of Eq. (3.5) $\alpha \equiv 0$ (or the trivial solution of system (3.3), respectively) is asymptotically stable (unstable).

In particular, when

$$\beta = H_1 \tag{3.6}$$

system (3.3) becomes conservative.

System (3.3) (like system (3.2)) has only four fixed points on the phase cylinder $S^1\{\alpha \bmod 2\pi\} \times R^1\{\omega\}$, which are specified by the relations

$$1)\ \alpha = \pi/2, \quad \omega = \omega_0, \quad \omega_0 = \beta/(1 + H_1); \quad 2)\ \alpha = -\pi/2, \quad \omega = -\omega_0$$
$$3)\ \alpha = 0, \quad \omega = 0;\ 4)\ \alpha = \pi, \omega = 0$$

We will classify the types of these fixed points.

For the point (0, 0) (and the point (π, 0)), the matrix of the linearized system has the form

$$\begin{Vmatrix} \beta & -(1+H_1) \\ \beta & -H_1 \end{Vmatrix} \quad \left(\text{respectively} \begin{Vmatrix} -\beta & -(1+H_1) \\ \beta & -H_1 \end{Vmatrix}\right)$$

enabling us to draw the conclusion that the point under investigation is attracting (repelling) when $H_1 > \beta$ and repelling (attracting) when $H_1 < \beta$ and that the point will be either a focus or a node, depending on the value of $(H_1 - \beta)^2 - 4\beta$.

Note that the vector field of system (3.2) has two types of symmetry: central symmetry and a so-called certain mirror symmetry[6,9] relative to straight lines (3.1). The types of the singular points $(-\pi/2, -\omega_0)$ and $(\pi/2, \omega_0)$ are ambiguous; therefore, after combining their investigation, we conclude that the matrix of the corresponding linearized system has the form

$$\begin{Vmatrix} 0 & 1 + H_1 \\ \beta - \dfrac{H_1\beta}{1 + H_1} & 0 \end{Vmatrix}$$

i.e., these points are saddle points.

We will study the structure of the split on the trajectory of system (3.3). For this purpose, we calculate the characteristic functions[9,10] of system (3.3) and of the system obtained from it after formally setting H_1 equal to zero:

$$\alpha' = -\omega + \beta\sin\alpha, \quad \omega' = \beta\sin\alpha\cos\alpha \tag{3.7}$$

This (ordered) characteristic function specifies the sign of the sine of the angle from the vector of the field of system (3.3) at each fixed point to the vector of the field of system (3.7) at the same point.[10]

System (3.7) is equivalent to the equation

$$\ddot{\alpha} - \beta\dot{\alpha}\cos\alpha + \beta\sin\alpha\cos\alpha = 0$$

The phase portrait of system (3.7) is shown in Fig. 2 (see also Refs. 6 and 10).

Knowing the structure of the phase portrait of system (3.7), we obtain information about the phase portrait of system (3.3).

The characteristic function of these systems

$$\chi = -H_1\omega\beta\sin\alpha\cos\alpha - H_1\omega^2\cos\alpha + H_1\omega\beta\sin\alpha\cos\alpha = -H_1\omega^2\cos\alpha \tag{3.8}$$

is sign-definite. When $H_1 \neq 0$, the sine of the angle cited above has the same sign almost everywhere in the band $\Pi = \{(\alpha, \omega) \in R^2: -\pi/2 < \alpha < \pi/2\}$, and it also has the same, but opposite sign in the band $\Pi' = \{(\alpha, \omega) \in R^2: \pi/2 < \alpha < 3\pi/2\}$.

In fact, by virtue of relation (3.8), the vector field of the reference system rotates in the same direction relative to the vector field of system (3.7) monotonically with respect to $H_1 > 0$. Therefore, it can be proved that the phase portrait of system (3.3) is of the same type as the phase portrait of system (3.7) when $0 < H_1 < \beta$, since system (3.3) is equivalent to system (3.7).[9,11]

Based on this fact, we can draw a conclusion regarding the structure of the phase portrait of system (3.3) for any β and H_1.

In general, dynamical systems with analytic right-hand sides that correspond to Chaplygin's functions, which specify the action of a medium on a rigid body, are mainly investigated in the present study. A complete, more rigorous investigation of the general case (Ref. 9) can be cited.

The phase portrait of the system for $H_1 > \beta$ is obtained from the phase portrait shown in Fig. 2 by displacing the vertical axis by π (to the right or the left). To prove this, it is sufficient to make the replacement $\alpha \to \alpha + \pi$ in the reference system. Therefore, when $H_1 > \beta$, the origin of coordinates becomes an asymptotically stable equilibrium position.

Under condition (3.6), system (3.3) takes the following form:

$$\alpha' = -\omega + \beta\sin\alpha - \beta\omega, \quad \omega' = \beta\sin\alpha\cos\alpha - \beta\omega\cos\alpha \tag{3.9}$$

Making the replacement of variables $u = \omega - \beta\sin\alpha + \beta\omega$ in system (3.13) and, specifically, "linearizing" its vector field along the curve $\{(\alpha, \omega) \in R^2: (1+\beta)\omega = \beta\sin\alpha\}$, we obtain a new system that is equivalent to system (3.9): $\alpha' = -u$, $u' = \beta\sin\alpha\cos\alpha$. This confirms that it is conservative.

System (3.3) (as is the more general system (3.2)) is a so-called system with variable dissipation with zero mean,[9] which has additional non-trivial non-linear properties. In such systems:

a) there are two types of symmetry (as was indicated above); central symmetry and a certain mirror symmetry of its vector field relative to the straight lines (3.1) (Fig. 2);
b) a phase volume with variable density is maintained in the region of the phase cylinder, which is entirely filled with rotational motions;
c) a transcendental first integral, which is expressed in terms of a finite combination of elementary functions under certain conditions, exists.

The last property was specially noted because in the further investigation of the spatial motion of a body, the high-order systems studied will have similar features.

We will mention one more property of such systems, which underlies the definition of the term "a system with variable dissipation with zero mean" (Ref. 9).

We will calculate the divergence $\mathrm{div}(\alpha, \Omega)$ of the right-hand side of system (3.2), which characterizes the variation of the area on its phase cylinder:

$$\mathrm{div}(\alpha, \Omega) = \frac{\sigma\upsilon}{I}\frac{d}{d\alpha}\frac{F(\alpha)}{\cos\alpha} - \frac{\sigma}{I}H\Omega\frac{d}{d\alpha}\frac{s(\alpha)}{\cos\alpha} - \frac{\upsilon}{I}Hs(\alpha)$$

This divergence is equal to zero on the average over a period with respect to the angle of attack, since clearly

$$\int_0^{2\pi} \mathrm{div}(\alpha, \Omega)d\alpha \equiv 0$$

4. The problem of a pendulum immersed in the flow of a medium

We will briefly analyse the problem of a physical pendulum in a flow.[11] This problem enable us to discover qualitative similarities in the dynamics of partially fixed and free bodies.

The model of the action of the flow of a medium on a fixed pendulum used is similar to the model constructed for a free body and takes into account the influence of the rotational derivative of the moment of the force exerted by the medium with respect to the angular velocity of the pendulum.

Consider the uniform flat plate *AB*, which is symmetrical about the plane perpendicular to the plane of the figure and passes through the support *OD*. The plate is rigidly fixed perpendicular to the support *OD*, which is located on the cylindrical hinge *Q*, and immersed in the uniform flow the medium (Fig. 3). In this case, the body is equivalent to a physical pendulum, in which plate *AB* and the hinge axis are perpendicular to the plane of motion. The medium flows with a constant velocity $\upsilon \pm \mathbf{0}$, and the support does not create friction.

We postulate that the total force **S** exerted by the medium on the plate is directed parallel to the support and that the point *N*, at which this force is applied, is specified by the angle of attack α, which is measured between the velocity vector υ_D of point *D* relative to the flow and the support *OD*, as well as the reduced angular velocity $\omega = l\Omega/\upsilon_D$ (l is the length of the support). Similar conditions arise when the model of jet flow past flat bodies is used.[3,11]

Thus, we will assume that the force **S** is directed along a normal to the plate in the direction opposite to the direction of the velocity $\boldsymbol{\upsilon_D}$ and passes through a point *N* on the plate that is displaced upstream from the point *D*.

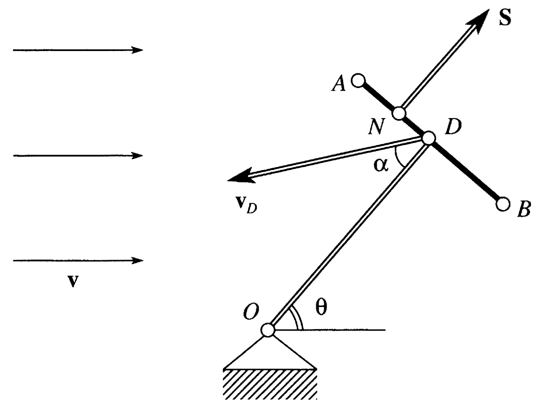

Fig. 3.

The vector $\mathbf{e}=\mathbf{OD}/l$ specifies the orientation of the support. Then $\mathbf{S}=s_1(\alpha)v_D^2\mathbf{e}$ ($s(\alpha)=s_1(\alpha)$ sign cos α, where the resistance coefficient s_1 depends only on the angle of attack). By virtue of the symmetry properties of the plate about the point D, the function $s(\alpha)$ satisfies condition (2.3).

If I is the central moment of inertia of the pendulum, Ω is the algebraic value of the projection of the angular velocity of the body onto the hinge axis, and $R(\alpha, l\Omega/v_D)$ is the distance from the centre of the plate D to the centre of pressure (the point N), the overall equation of motion of the pendulum takes the following form (θ is the angle of deflection of the pendulum, $\theta^\bullet=\Omega$)

$$I\Omega^\bullet = R(\alpha, l\Omega/v_D)s(\alpha)v_D^2 \tag{4.1}$$

Here we choose the functions that conform to the action of the medium R and s in the following form

$$R(\alpha, l\Omega/v_D) = y_N = A\sin\alpha - h_1^*\Omega/v_D,\ \ s(\alpha) = B\cos\alpha, \quad A, B, h_1^* > 0 \tag{4.2}$$

If we add the kinematic relations $v_D\cos\alpha=-v\cos\theta$ and $v_D\sin\alpha=l\Omega+v\sin\theta$ (here $v_D^2\cos\alpha\sin\alpha=-v^2\cos\theta\sin\theta-lv\Omega\cos\theta$), the fundamental equation (4.1) takes the form

$$I\Omega^\bullet = ABv_D^2\sin\alpha\cos\alpha - h_1\theta^\bullet v_D\cos\alpha = AB[-v^2\cos\theta\sin\theta - lv\theta^\bullet v_D\cos\alpha] + h_1 v\cos\theta,$$

$$h_1 = h_1^* B$$

Thus, when the functions (4.2) are chosen, the equation of motion of the pendulum has the form

$$I\theta^{\bullet\bullet} + (ABl - h_1)v\theta^\bullet\cos\theta + ABv^2\sin\theta\cos\theta = 0 \tag{4.3}$$

and is equivalent to Eq. (3.5) (and therefore to system (3.3)).

As we see, the parameter h_1 plays a key role in resolving the question of the stability of the solution $\theta\equiv\pi$ of Eq. (4.3). When the damping is fairly large ($h_1 > h_* = ABl$), such a solution becomes asymptotically stable in Lyapunov's sense.

The last equality is similar to Eq. (3.6) for a free body.

5. Statement of the problem of the three-dimensional motion of an axisymmetric body in a resistive medium

We will consider the problem of the three-dimensional motion of a uniform axisymmetric rigid body of mass m, part of whose surface has the shape of a flat circular disk that interacts with the medium according to the laws of jet flow (see Refs. 7 and 12)[2]. Let the remainder of the body surface lie within the volume bounded by the jet surface extending from the edge of the disk and not experience the action of the medium. Similar conditions can occur, for example, after the end of uniform circular cylinders enter water.[3]

We will assume that there are no forces tangential to the disk. Then the force $\mathbf{S}$ exerted by the medium on the body at the point N does not vary in orientation relative to the body (it is directed along a normal to the disk) and is quadratic with respect to the velocity of its centre D (Newtonian resistance, Fig. 4). It is also assumed that the gravitational force acting on the body is negligibly small compared with the resistive force (action) of the medium.

When the above conditions hold, among the motions of the body there is a regime of rectilinear translational deceleration (unperturbed motion), similar to the case of plane-parallel motion: the body is capable of performing translational motion in the direction of its axis of symmetry, i.e., perpendicular to the plane of the disk.

We connect the right-hand system of coordinates $Dxyz$ to the body (Fig. 4) and direct the Dx axis along the axis of symmetry of the body. We rigidly connect the Dy and Dz axes to the disk. The components of the angular velocity vector $\boldsymbol{\Omega}$ in the $Dxyz$ system will be denoted by Ω_x, Ω_y and Ω_z. The moment of inertia tensor of a dynamically symmetrical body in the $Dxyz$ axes has the form diag$\{I_1, I_2, I_1\}$.

[2] See also: Samsonov VA, Shamolin MV. Deceleration of a body in a medium under jet flow. *Nauchn Otchet Inst Mekh Mosk Gos Univ* 1991;4141.

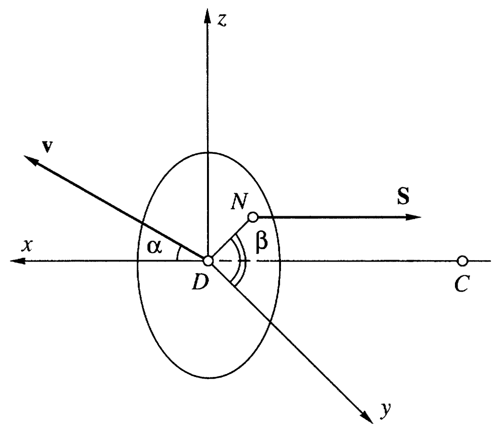

Fig. 4.

We will utilize the quasistationary-state hypothesis, and, for simplicity, we will assume that the quantity $R = DN$ is determined by at least the angle of attack α, i.e., by the angle between the velocity vector υ of the centre D and the straight line Dx. Thus, $DN = R(\alpha, \ldots)$.

In addition, we will take the magnitude of the resistance force **S** in the form $S = s_1(\alpha)\upsilon^2$, where $\upsilon = |\upsilon|$. For convenience in the ensuing description (as in the case of plane-parallel motion), instead of the drag coefficient $s_1(\alpha)$, we will introduce the auxiliary sign-variable function $s(\alpha)$: $s_1 = s_1(\alpha) = s(\alpha)$ sign $\cos\alpha \geq 0$. Thus, the pair of functions $R(\alpha, \ldots)$ and $s(\alpha)$ specifies the characteristics of the action of the medium on the disk.

Let us consider the spherical coordinates $(\upsilon, \alpha, \beta_1)$ of the tip of the velocity vector $\upsilon = \upsilon_D$ of point D relative to the flow, in which the angle β_1 is measured in the plane of the disk (Fig. 4). Using unintegrable relations to express υ, α and β_1 in terms of cyclic kinematic variables and velocities, we can treat them as quasivelocities after adding to them the components of the angular velocity $(\Omega_x, \Omega_y, \Omega_z)$ in the $Dxyz$ axes, in which clearly $\upsilon_D = \{\upsilon\cos\alpha, \upsilon\sin\alpha\cos\beta_1, \upsilon\sin\alpha\sin\beta_1\}$.

By virtue of the theorems of motion of the centre of mass (in projections onto the associated $Dxyz$ axes) and the variation of the angular momentum, we obtain the dynamical part of the differential equations of motion, which can be treated in a six-dimensional phase space of the quasivelocities ($\sigma = DC$):

$$\upsilon^{\cdot}\cos\alpha - \alpha^{\cdot}\upsilon\sin\alpha + \Omega_y\upsilon\sin\alpha\sin\beta_1 - \Omega_z\upsilon\sin\alpha\cos\beta_1 + \sigma(\Omega_y^2 + \Omega_z^2) = -s(\alpha)\upsilon^2/m$$

$$\upsilon^{\cdot}\sin\alpha\cos\beta_1 + \alpha^{\cdot}\upsilon\cos\alpha\cos\beta_1 - \beta_1^{\cdot}\upsilon\sin\alpha\sin\beta_1 + \Omega_z\upsilon\cos\alpha -$$

$$-\Omega_x\upsilon\sin\alpha\sin\beta_1 - \sigma\Omega_x\Omega_y - \sigma\Omega_z^{\cdot} = 0$$

$$\upsilon^{\cdot}\sin\alpha\sin\beta_1 + \alpha^{\cdot}\upsilon\cos\alpha\sin\beta_1 + \beta_1^{\cdot}\upsilon\sin\alpha\cos\beta_1 + \Omega_x\upsilon\sin\alpha\cos\beta_1 -$$

$$-\Omega_y\upsilon\cos\alpha - \sigma\Omega_x\Omega_z + \sigma\Omega_y^{\cdot} = 0$$

$$I_1\Omega_x^{\cdot} = 0, \quad I_2\Omega_y^{\cdot} + (I_1 - I_2)\Omega_x\Omega_z = -z_N s(\alpha)\upsilon^2, \quad I_2\Omega_z^{\cdot} + (I_2 - I_1)\Omega_x\Omega_y = y_N s(\alpha)\upsilon^2 \tag{5.1}$$

Here y_N and z_N are the Cartesian coordinates of the point of application N of the resistance force in the plane of the disk.

6. The three-dimensional motion of a body in a resistive medium in the presence of a follower force

We will single out the class of problems of the action of a medium on a body in which a follower force acts along its geometric axis of symmetry (compare the case of plane-parallel motion). Under certain conditions, this force ensures the realization of classes of motions (superimposed constraints) that are of interest. In this case, the follower force is the reaction of these constraints. When there is no follower force, the body undergoes free spatial deceleration in the resistive medium (see also Ref. 12). In this study, the follower force ensures that condition (1.2), is satisfied at all times.

By virtue of equalities (5.1), the Routh cyclic invariant relation $\Omega_x = \Omega_{x_0} = \text{const}$ holds at all times.

In the following we will investigate the case of zero rotation of the body about axis of its symmetry, i.e., the case where the following condition holds: $\Omega_{x_0} = 0$.

Then the independent dynamical part of the equations of motion in four-dimensional phase space has the form

$$\alpha^{\cdot}\upsilon\cos\alpha\cos\beta_1 - \beta_1^{\cdot}\upsilon\sin\alpha\sin\beta_1 + \Omega_z\upsilon\cos\alpha - \sigma\Omega_z^{\cdot} = 0$$

$$\alpha^{\cdot}\upsilon\cos\alpha\sin\beta_1 + \beta_1^{\cdot}\upsilon\sin\alpha\cos\beta_1 - \Omega_y\upsilon\cos\alpha + \sigma\Omega_y^{\cdot} = 0$$

$$I_2\Omega_y^{\cdot} = -z_N s(\alpha)\upsilon^2, \quad I_2\Omega_z^{\cdot} = y_N s(\alpha)\upsilon^2 \tag{6.1}$$

System (6.1) contains the functions y_N, z_N and s of the action of the medium, which will be described qualitatively using experimental information on the properties of jet flow.

For simplicity, we will confine ourselves to an investigation of system (6.1) for the following functions of the action of the medium (we will also call the system obtained here a "reference" system)

$$y_N = A\sin\alpha\cos\beta_1 + h\Omega_z/\upsilon,$$
$$z_N = A\sin\alpha\sin\beta_1 - h\Omega_y/\upsilon, \quad s(\alpha) = B\cos\alpha$$
$$A, B, h > 0 \tag{6.2}$$

In equalities (6.2) the coefficient h appears in front of terms that are proportional to the rotational derivatives of the moment of the force exerted by the medium with respect to the components of the angular velocity of the body.[13,14]

System (6.1) is a dynamical system with variable dissipation with zero mean (with respect to the angle of attack). This means that the integral of the divergence of its right-hand side with respect to the period of the angle of attack, which corresponds to the variation of the phase volume (after some reduction of the system), is equal to zero. The system is "semiconservative" in a certain sense.

Next, projecting the angular velocities onto the mobile axes that are not bound to the body so that $z_1 = \Omega_y\cos\beta_1 + \Omega_z\sin\beta_1$ and $z_2 = -\Omega_y\sin\beta_1 + \Omega_z\cos\beta_1$ and introducing the dimensionless variables w_k ($k = 1, 2$) and the parameters defined by the formulae $\sigma z_k = \upsilon w_k$, $h_1 = hB$, $H_1 = \sigma h_1/I_2$ and $\beta = \sigma^2 AB/I_2$ (here $\alpha' = \alpha^\bullet\,\upsilon/\sigma$ etc.), we obtain the following fourth-order analytic system

$$\alpha' = -(1+H_1)w_2 + \beta\sin\alpha \tag{6.3}$$
$$w_2' = \beta\sin\alpha\cos\alpha - (1+H_1)w_1^2\frac{\cos\alpha}{\sin\alpha} - H_1w_2\cos\alpha, \quad w_1' = (1+H_1)w_1w_2\frac{\cos\alpha}{\sin\alpha} - H_1w_1\cos\alpha$$
$$\beta_1' = (1+H_1)w_1\frac{\cos\alpha}{\sin\alpha} \tag{6.4}$$

It contains the independent third-order subsystem (6.3).

When $\beta = H_1$, the divergence of the right-hand side of system (6.3) ((6.3), (6.4)) is identically equal to zero after the replacement of variables $w_1^* = \ln|w_1|$, which enables us to consider this system (systems) to be conservative.

Theorem 1. System (6.3), (6.4) has a complete set of first integrals, which are elementary transcendental functions of their phase variables. Two of them form a complete set of first integrals of system (6.3).

In this case their transcendentality is understood in the sense of the theory of functions of a complex variable: after a function is extended into the complex region, it has essential singularities.[9]

In fact, after making the substitution $\tau = \sin\alpha$, we can make system (6.3) correspond to the following non-autonomous second-order system

$$\frac{dw_2}{d\tau} = \frac{\beta\tau - (1+H_1)w_1^2\tau^{-1} - H_1w_2}{-(1+H_1)w_2 + \beta\tau}, \quad \frac{dw_1}{d\tau} = \frac{(1+H_1)w_1w_2\tau^{-1} - H_1w_1}{-(1+H_1)w_2 + \beta\tau} \tag{6.5}$$

Next, introducing the replacement $w_k = u_k\tau$ ($k = 1, 2$), which is characteristic of homogeneous systems, we can match system (6.5) to a non-autonomous differential equation of the form $du_2/du_1 = f(u_1, u_2)$, which has a first integral. Hence it follows that system (6.3) has the first integral

$$[(1+H_1)w_2^2 - (H_1+\beta)w_2\sin\alpha + (1+H_1)w_1^2 + \beta\sin^2\alpha]/(w_1\sin\alpha) = C_1 \tag{6.6}$$

As was already noted, when $\beta = H_1$, the dynamical system (6.3) (like system (6.3), (6.4)) is conservative. In fact, when condition (3.6) holds, relation (6.6) transforms to the following invariant relation

$$[w_2^2 + (1+\beta)w_1^2 + \beta[w_2 - \sin\alpha]^2]/(w_1\sin\alpha) = C_1 \tag{6.7}$$

In addition, it is easy to verify that the numerator and denominator on the left-hand side of equality (6.7) are first integrals of system (6.3) when $\beta = H_1$. In the case when $\beta \pm H_1$, system (6.3), (6.4) ceases to be conservative. Then neither the numerator nor the denominator of invariant relation (6.6) is a first integral. It is not necessary to verify this analytically because system (6.3) has attracting and repelling limit sets, which prohibit the presence of a complete set even of only continuous first integrals in the system under investigation throughout the entire phase state.[6,9]

The additional first integral of system (6.3) is found from the following quadrature formula

$$\int\frac{d\tau}{\tau} = \int\frac{\beta - (1+H_1)u_2}{\beta - (H_1+\beta)u_2 + (1+H_1)[u_2^2 - U(u_1, C_2)]}du_2 \tag{6.8}$$

where

$$U(u_2, C_1) = \frac{1}{2(1+H_1)}\left\{C_1 \pm \sqrt{C_1^2 - 4D_1}\right\}, \quad D_1 = (1+H_1)u_2^2 - (H_1+\beta)u_2 + \beta$$

at the level $C_1 > 4(1 + H_1)D_1$ of integral (6.7).

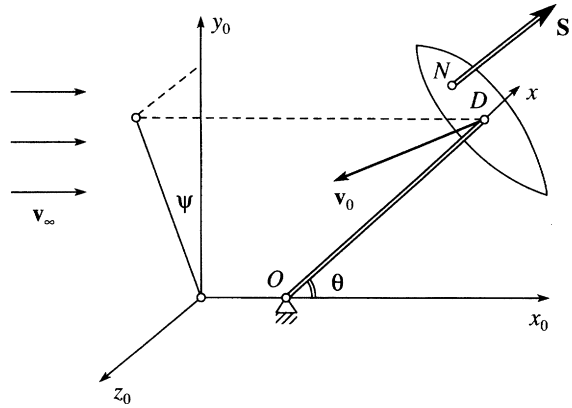

Fig. 5.

The general structural form of the additional first integral of system (6.3), (6.4) is as follows:

$$\Phi_1(w_1, w_2, \sin\alpha) = C_2$$

Here the constant C_2 appears after integration of relation (6.8).

By virtue of Eq. (6.4), the additional first integral of the fourth-order system (6.3), (6.4) is found from the solution of the equation

$$\frac{du_1}{d\beta_1} + \frac{\beta - (1 + H_1)u_2}{1 + H_1} = u_2 - \frac{H_1}{1 + H_1}$$

which leads to the invariant relation sought (C_3 is a constant of integration)

$$\sin^2\{2(1 + H_1)^2(\beta_1 + C_3)\} = \frac{(2(1 + H_1)w_1 - 2C_1\sin\alpha)^2}{[(H_1 + \beta)^2 - 4\beta(1 + H_1) + C_1^2]\sin^2\alpha}$$

7. The problem of a spherical pendulum immersed in the flow of a medium

By analogy with a free body, we will consider the problem of the motion of a spatial pendulum in the uniform flow of a medium; the flow acts only on the circular disk, whose centre is rigidly fastened perpendicular to a rigid support, whose other end is fastened to a spherical hinge. The model of the action of the medium on the disk is the same as in the previous problem (Fig. 5).

We will consider the motion of such a pendulum in the flow of a medium without twisting of the pendulum itself ($\Omega_{x0} = 0$). As before, the dependence of the moment of force exerted by the medium on the angular velocity of the body is taken into account in the case of Chaplygin's functions that specify the action of the medium and lead to equalities (6.2).

The equations of motion of such a pendulum have the form

$$\Omega_y^\cdot = -\frac{1}{I_2}\upsilon_D^2 R\left(\alpha, \frac{l\Omega}{\upsilon_D}\right)s(\alpha)\sin\beta_1, \quad \Omega_z^\cdot = \frac{1}{I_2}\upsilon_D^2 R\left(\alpha, \frac{l\Omega}{\upsilon_D}\right)s(\alpha)\cos\beta_1$$

The functions R and s satisfy conditions (4.2).

Let θ and ψ be angles that specify the position of the spatial pendulum on the sphere S^2 (Fig. 5). We will measure the angle θ from the x_0 axis to the support, and we will measure ψ from the projection of the support onto the $0y_0z_0$ plane to the y_0 axis (we assume that $\psi = 0$ at the initial instant). Then the equalities that relate (υ_D, α, β_1) and (θ, ψ, Ω_y, Ω_z), where l is the length of the support, have the form

$$\upsilon_D\cos\alpha = -\upsilon_\infty\cos\theta, \ \upsilon_D\sin\alpha\cos\beta_1 = l\Omega_z + \upsilon_\infty\sin\theta\cos\psi,$$
$$\upsilon_D\sin\alpha\sin\beta_1 = -l\Omega_y - \upsilon_\infty\sin\theta\sin\psi$$

and, by virtue of kinematic relations that are similar to the kinematic Euler formulae,

$$\Omega_y = \theta^\cdot\sin\psi + \psi^\cdot\operatorname{tg}\theta\cos\psi, \quad \Omega_z = \theta^\cdot\cos\psi - \psi^\cdot\operatorname{tg}\theta\sin\psi$$

Then the equations of motion of such a system on the tangential bundle T_*S^2 of the two-dimensional sphere can be represented in the following form

$$\theta^{\cdot\cdot} + (\tilde{\beta} - \tilde{H}_1)\theta^{\cdot}\cos\theta + \tilde{\beta}\sin\theta\cos\theta - \psi^{\cdot 2}\frac{\sin\theta}{\cos\theta} = 0$$

$$\psi^{\cdot\cdot} + (\tilde{\beta} - \tilde{H}_1)\psi^{\cdot}\cos\theta + \theta^{\cdot}\psi^{\cdot}\frac{1+\cos^2\theta}{\cos\theta\sin\theta} = 0 \tag{7.1}$$

Here $\beta = l^2AB/I_2$ and $H_1 = lBh/I_2$ are dimensionless physical constants, and, as before, the coefficient H_1 is proportional to the rotational derivatives of the moment of hydrodynamic or aerodynamic forces with respect to the components of the angular velocity of the spatial pendulum. The length of the support is equivalent to the distance σ, and the constant velocity of the incident flow is equivalent to the constant parameter υ. In addition, the angle of attack for a free body is equivalent to the angle of deflection θ of the pendulum from the velocity vector of the flow, and the angle β_1 is equivalent to the cyclic variable, i.e., to the angle ψ.

Theorem 2. System (6.3) is equivalent to system (7.1).

Note that in the case where $\cos\theta = 0$, system (7.1) can be extended by continuity, and the singularity $\sin\theta = 0$ is purely kinematic, since the spherical coordinates under consideration $(\upsilon, \alpha, \beta_1)$ degenerate at it.

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